DEVELOPING A TOOL FOR MEASURING STUDENT ORIENTATIONS WITH RESPECT TO UNDERSTANDING IN MATHEMATICAL LEARNING

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The goal of this paper is twofold. First, the paper clarifies and elaborates on an important theoretical construct called orientation with respect to understanding in mathematics, which denotes the degree to which students exhibit an inclination towards and demonstrate an earnest concern for understanding in mathematical learning. Second, the paper reports on the creation and evaluation of a methodological tool for measuring the aforementioned construct. The tool was operationalized from analyses of 38 college students' problem-solving behaviors as well as their verbal self-reflections in semi-structured task-based interviews. Results showed decent validity and reliability evidence on the proposed research tool. This study contributes to a better conceptualization of learning orientation as a fundamental shaper of how students engage with mathematics; it also holds practical potential for enhancing mathematics classrooms.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Measurement; Research Methods

Helping students understand is one of the most important goals in mathematics education (Common Core State Standards, 2010; Schoenfeld et al., 2023). Literature on understanding provides well-articulated conceptualizations about how people develop understanding (Piere & Kieren, 1989, 1994) as well as what understanding is from a psychological perspective (Hiebert & Carpenter, 1992; Sierpinska, 1994), performance-based perspective (Perkins & Blythe, 1994), sociocultural perspective (Godino, 1996; Johnson, 1987), and affective perspective (Duffin & Simpson, 2000). Yet there is an under-addressed challenge in precisely explaining students' different understanding-related behaviors and using it to inform classroom teaching and learning. For example, what accounts for some students' consistent inclination to memorize and apply formulas without necessarily making sense of them, as compared to others' tendency to scrutinize the underlying mathematical logic? In an attempt to articulate the difference more precisely and investigate its potential impact on student learning, I extend Schoenfeld's (2010) definition and define *orientation* as including people's goals, attitudes, values, dispositions, beliefs, and preferences. I further define students' *orientations with respect to understanding* to be the degree to which students exhibit an inclination towards and demonstrate an earnest concern for understanding in mathematical learning. Students with a strong orientation towards understanding care deeply about how things work, prioritize goals and actions for developing understanding, and are willing to put in much effort to understand underlying principles.

Theoretical Background

The construct of student orientations with respect to understanding is important and is theoretically consistent with existing literature. Schoenfeld's (2010) theory on goal-oriented decision making argues that the key determinants of human behaviors are resources (including knowledge, heuristics and other perceived resources), goals (their conscious or unconscious aims or objectives), and orientations (including their beliefs, values, dispositions, and biases). This suggests a fundamental relationship between student orientations and their behaviors, providing a theoretical foundation that the more we want to understand how students are engaged in meaningful learning and problem solving, the more we need to know about their orientations with

respect to understanding. Moreover, the construct appears to significantly correlate with (if not explain) students' mathematical achievements, as Mashaal (2006) reports that lower-achieving students tend to rely on memorization strategies while higher-achieving students tend to put in more effort to understand mathematical concepts and principles. Since the present body of literature has not rigorously examined the construct of student orientations with respect to understanding (Schoenfeld, personal communication, January 24, 2023), it is imperative that we operationalize it and develop a research tool to support more sophisticated discussions about students' orientations in relation to their mathematical learning behaviors and outcomes.

There are indeed works from Dweck and colleagues that suggest a type of orientation in particular to goals and mindsets is central to student learning. According to the achievement goals framework (Dweck, 1986; Maehr, 1984; Nicholls, 1984), students' performance-oriented goals seek to outperform others and to demonstrate comparative competencies (Elliot, 2005), and mastery-oriented goals emphasize the desire to "understand a task, acquire new knowledge, and develop abilities" (Darnon, Butera & Harackiewicz, 2007, p. 61). Performance-oriented goals are often assumed to result in undesirable outcomes (such as poor achievement, avoidance of challenges, anxiety) and their benefits are limited to simple tasks (Dweck, 1986; Nicholls, 1984; Senko, 2019). Mastery-oriented goals, in contrast, have been found to improve student engagement and perseverance in challenging tasks and are believed to be conducive to high achievements, openness to collaboration, effective study strategies, and students' well-being (Senko, 2019). According to Dweck (2006, 2012), how students perceive their abilities also play an important role in their motivation and achievement. More specifically, those who believe that intelligence can be learned through effort are said to have a "growth mindset" (or an "incremental" belief) and those who believe that intelligence is fixed, and someone is either smart or not are said to have a "fixed mindset" (or an "entity" belief). Literature shows that students with a growth mindset are more likely to exert effort to overcome challenges and display greater resilience when encountering failure (Blackwell et al., 2007; Dweck & Leggett, 1988; Dweck & Yeager, 2019); students with a fixed mindset are more prone to evading challenges and conceding to setbacks (Dweck, 2007a, 2007b, 2013).

The Pirie-Kieren model for growth of mathematical understanding (Pirie & Kieren, 1989, 1994) and Huang's (2022) observation system provide useful insights for understanding potential indicators of students' orientations with respect to understanding in mathematical learning. In Pirie-Kieren's works, growth in understanding is seen as a dynamic, active, non-linear, and transcendently recursive process involving the building of and acting in the world (Pirie & Kieren, 1991; Pirie & Martin, 2000). The metaphor of recursion is fundamental to the Pirie-Kieren model, and the term *fold back* is defined specifically to describe the phenomenon of returning to an initial understanding, reflecting on and reorganizing earlier conceptions, and effectively building a thicker understanding in response to complex problem situations (Pirie & Martin, 2000, p. 131). Huang (2022) proposes an observation system that characterizes students' understanding-oriented behaviors in the context of problem solving, suggesting that students' orientation with respect to understanding indeed matters and that it is correlated with students' problem-solving performances. Huang (2022) further argues that the behavioral indicators of (a) problematizing ongoing works, (b) suggesting solving with multiple approaches, (c) examining proposed ideas, and (d) examining problem statements give rise to promising profiles of students' orientations with respect to understanding. Collectively, existing literature offers a theoretically coherent and rich foundation upon which I seek to push the conversation further by developing a more refined research tool.

The Refined Construct: Orientation with Respect to Understanding

I develop a taxonomy of four qualitatively different levels in the continuum of students' orientations with respect to understanding in mathematical learning. At the highest level, *proactive to understanding* (PU), students proactively seek understanding regardless of the learning and problem-solving situations; they care deeply about how things work and are willing to put in much effort to understand underlying principles. At the second level, *open to understanding* (OU), students typically make an effort to understand mathematical ideas under most circumstances. Students at this level are open to developing a deeper understanding of mathematics; they may seek understanding when prompted, but they may not take initiative in doing so. At the third level, *indifferent to understanding* (IU), students do not have a strong preference for developing a deep understanding of mathematics; they may resort to memorization most of the time. Students at this level are generally comfortable with memorizing formulas, although they do make an effort to understand complex mathematical concepts on occasion. At the lowest level, *resistant to understanding* (RU), students resist understanding by ignoring their confusions, shutting down conflicting ideas, or blindly applying formulas or procedures without making sense of them. Table 1 summarizes the levels with descriptions and examples operationalized from my analyses of 38 college students' behaviors and reflections.

Table 1: Orientation with Respect to Understanding

Levels	Descriptions	Examples
Proactive to Understanding (PU)	Care deeply about how things work and are willing to put in much effort to understand underlying principles; proactively seek understanding in learning and problem solving.	Spend time understanding confusing ideas even after getting an answer; problematize ongoing works ("What if …?"); proactively suggest solving a problem with multiple approaches.
Open to Understanding (OU)	Make an effort to understand mathematics under most circumstances; seek a deeper understanding when prompted but may not take initiative in doing so.	Fold back (Pirie & Kieren, 1991) to initial understanding when confronted with challenges; examine conflicting ideas; probe further clarification or justification from their peers.
Indifferent to Understanding (IU)	Do not have a strong preference for developing a deep understanding of mathematics; resort to memorization most of the time.	Watch their peers sort out confusions with no evidence of contributing; only reexamine the problem statement when prompted.
Resistant to Understanding (RU)	Not willing to put in effort for understanding; want to memorize and apply formulas or procedures without making sense of them.	Ignore or shut down conflicting ideas; ignore their own confusions; blindly apply formulas or procedures without examining their applicability.

Methods

Item Design

An overarching assumption underlying my work is that what students say is not always the same as what they actually think or believe in, positioning their learning behaviors in actual problem situations as potentially more reliable measurement targets than their self-assessments or verbal reports. On the other hand, students' reflections about their own learning could offer insights about their orientations that might not be otherwise evident from purely examining their behaviors. Thus, my work seeks to operationalize the construct of student orientations with respect

to understanding using both students' behaviors and their verbal self-reflections as a triangulation (Thurmond, 2001) strategy. I designed a semi-structured task-based interview (Ginsburg, 1997; Goldin, 2000) and asked students to (a) think aloud and try to solve two challenging mathematical problems (so as to elicit behaviors that might indicate certain level of orientations), and (b) reflect on their mathematical learning experiences (so as to elicit what they actually think about understanding and how they approach it on a daily basis).

Problem-Solving Tasks. This item was built on the assumption that challenging problem situations afforded distinction between students at different levels of the continuum of learning orientations: If students encountered a challenging problem that they did not immediately know how to solve, some might work hard to understand relevant mathematics and build a more sophisticated understanding, while others might try to plug in (not-necessarily related) formulas they remembered without necessarily making sense of the situation or the mathematics. Below is one example task that I used to elicit college students' orientations in a problem situation.

> You have 100 pennies ($0.01) on a table. Abby comes in and replaces every second penny with a nickel ($0.05), which means the 2nd, 4th, ..., 100th coins become nickels. Oliver then replaces every third coin with a dime ($0.10), which means now the 3rd, 6th, ..., 99th coins become dimes. After that, Emma replaces every fourth coin with a quarter ($0.25), which means now the 4th, 8th, ..., 100th coins become quarters. Summing all the 100 coins on the table, how much money will you have?

Open-Ended Interview Questions. To zoom out of the problem-solving situation, I developed an interview protocol to explore students' orientations in their general mathematical learning experiences. Two personal-reflection item bundles were particularly relevant: one investigated students' treatments to understanding versus memorizing, and the other investigated students' behaviors and their reasoning behind when they did not understand something about mathematics. The first bundle (I_1) prompted the following: (a) "Reflecting on your mathematical experience in classrooms, when do you make an effort to understand mathematical ideas, and when would you be okay with just memorizing formulas or procedures?", and (b) "Reflecting on your experience feeling stuck in mathematical problems, when would you make an effort to go back and try to understand the underlying mathematics, and when would you be okay with recalling and applying formulas without evaluating whether they are applicable?". The second bundle (I_2) asked students to reflect on and describe the last time they did not understand something about mathematics, with a focus on what they did in the situations.

Accounting for the External Variable of Math Anxiety. Among other things, students' math anxiety (Ashcraft, 2002) might affect the extent to which students were oriented towards understanding in mathematical learning. I hypothesized that if students felt comfortable learning and doing mathematics, they were more likely to try and make sense of underlying concepts and principles; if students had high levels of math anxiety, they were more likely to refrain from confusions or challenging mathematical ideas. For this reason, I drew on May's (2009, p. 75) math anxiety questionnaire and added a subsection at the end of the interview to explore students' levels of math anxiety. The questions I asked were (A_1) "To what extent do you feel stressed in your math class?", (A_2) "To what extent do you worry that you will not be able to understand the math you learn?", and (A_3) "To what extent is doing math stressful?".

Data Collection

I used semi-structured task-based interviews (Ginsburg, 1997; Goldin, 2000) along with think-aloud protocols (Leighton, 2017; Schoenfeld, 1985) to conduct this study. 19 self-selected pairs of

friends (defined by feeling comfortable discussing math together) from UC Berkeley volunteered to participate sequentially in Summer 2022. The participants took about 50-90 minutes to complete the problem-solving tasks as well as the subsequent semi-structured interview in pairs. I video-recorded all participation and collected all participants' written works after they completed the study. Among the 38 participants, 21 (55.3%) were female and 17 (44.7%) were male. 12 (31.6%) were STEM undergraduates, nine (23.7%) were non-STEM undergraduates, five (13.2%) were undergraduates with undeclared majors, eight (21.1%) were non-STEM graduate students, and four (10.1%) were STEM graduate students.

Data Analysis

I approached data analysis in five phases: (1) holistically reviewed the video-recordings of students' problem solving as well as their interview reflections to develop an overview of the data, (2) developed analytic memos to record my critical examination of each recording along with the corresponding time and the reasons I hypothesized to be potential indicators of student orientations, (3) identified emerging themes from the analytic memos and mapped each theme to the most reasonable level, (4) developed transcripts with high precision that captured a detailed account of students' problem solving behaviors as well as their verbal reflections and coded them, (5) took another holistic review of the recordings to evaluate whether the preliminary analysis provided a full story, compared it with my initial impression of the data, and revised the coding rubric accordingly. The results informed the revision of my proposed taxonomy (Table1).

The description column in Table 1 (which was iteratively revised in the data analysis process) provided an overarching coding rubric for the interview responses. Building on Pirie & Kieren's (1989, 1991, 1994) as well as Huang's (2022) work, I operationalized some behavioral indicators of each level for the problem-solving tasks. Student behaviors were coded as *proactive to understanding* if they showed evidence of at least one of the following: (a) spent time understanding confusing ideas even after getting a correct answer; (b) problematized their ongoing works (e.g., "What if ...?"), (c) proactively generalized problem-solving strategies, and (d) proactively suggested solving a given problem with multiple approaches. Students were coded as *open to understanding* if they (a) folded back (Pirie & Kieren, 1991) to prior knowledge and elaborated on or reconstructed incomplete understandings when confronted with challenges, (b) carefully examined conflicting ideas, or (c) probed further clarification or justification from their peers. Students were coded as *indifferent to understanding* if they (a) watched their peers sort out confusions with no evidence of contributing or (b) only reexamined (clarified, reread, or rephrased) the problem statement when prompted. Students were coded as *resistant to understanding* if they (b) ignored or shut down emergent, conflicting ideas, (c) ignored their own confusions, or (c) blindly applied formulas or procedures without examining their applicability. With the operationalized rubric, I used the BEAR Assessment System Software (Fisher & Wilson, 2019) to evaluate the item fit statistics, reliability, and validity of my proposed measure. I further created a scatterplot to investigate potential correlations between students' estimated orientations and their levels of math anxiety.

Results

Item Fit Statistics

I used the Rasch model (Bond & Fox, 2013) and item fit statistics to evaluate the proposed instrument and the extent to which the observed data fitted the model-implied distribution. Infit is an inlier-sensitive or information-weighted fit statistic, which measures the difference between the discrimination of an item and the average discrimination of other items in an instrument (Wu & Adams, 2013). Outfit is an outlier-sensitive fit statistic, which is a measure sensitive to unexpected

responses such as incorrect answers about easy questions by a high-performing student (Waterbury, 2020). The expected value of both Infit and Outfit is 1, with smaller values indicating an over-fit of the data to the Rasch model (i.e., responses are too predictable or Guttman-like) and greater values indicating an under-fit of the data (i.e., responses are not as predictable as expected). In applied Rasch measurement, it is common to flag items with fit-index values less than 0.75 and values greater than 1.33 (Blum et al., 2020). According to Wilson (in press), high fit-index values are often more problematic, suggesting the relevant item might measure something different from the intended construct. As shown in Table 2, all items except A_2 showed decent Infit and Outfit values; the Infit and Outfit values of A_2 were slightly smaller than 0.75 (meaning responses were "too predictable"), which might have been because A_2 was a reliable item for measuring students' levels of math anxiety (May, 2009).

Table 2: Item Fit Statistics

	Problem Solving Tasks	I_1	I_2	A_1	A_2	A_3
Infit	1.28	0.86	0.82	1.02	0.73	0.98
Outfit	1.16	0.87	0.88	0.88	0.71	0.99

Reliability

I implemented two types of reliability assessment for my measure. First, I calculated the internal consistency coefficient using the BEAR Assessment System Software (Fisher & Wilson, 2019): the EAP reliability was 0.909, the WLE reliability was 0.890, and the variance/covariance was 8.711. Second, I measured the inter-rater reliability using the classical test theory approach. I was lucky to have a volunteer from my undergraduate research mentees who independently coded the data using the rubric I provided. The percentage agreement between her and my initial code was 79.6%. All code conflicts were resolved, and the resulting discussions contributed to the revised coding rubric as shown in the Data Analysis section.

Validity

Content Validity. I drew from Schoenfeld's theory on decision making (2010) as well as Dweck's and colleagues' works (e.g., Dweck, 1986, 2006, 2012; Nicholls, 1984) to develop my proposed taxonomy of student orientations with respect to understanding. I provided a detailed description of the levels and a coding rubric with examples building on Pirie & Kieren's (1989, 1991, 1994) model, Huang's (2022) observation system, as well as careful operationalization from my data. I further calibrated my proposed measurement using the BEAR Assessment System Software (Fisher & Wilson, 2019) and the feedback I received from my informants.

Response Process Validity. I used think-aloud protocols (Leighton, 2017; Schoenfeld, 1985) to encourage the participants share their thinking process while completing the problem-solving tasks, so that I could triangulate their problem-solving behaviors using their spontaneous utterances, body movements, and the flow of mathematical ideas. At the end of the post-task interview, I asked the participants to evaluate which tasks or questions, if any, they had trouble understanding. The result was that all participants found the materials to be very accessible.

Internal Structure Validity. I used a WrightMap by Items (Figure 1) and a WrightMap by Levels (Figure 2) to evaluate the structural validity and consistency within my proposed measurement. According to Figure 1, the first thresholds (marked blue) of all items were generally below the second thresholds (marked green), which were all below the third threshold (marked red). This phenomenon is called banding, which implies that it was the levels (PU, OU, IU, RU) of

orientations as opposed to the items that appeared to primarily determine the threshold locations. This supports the hypothesis that the proposed levels were distinct and that the approach of mapping different levels of student orientations within different items was valid.

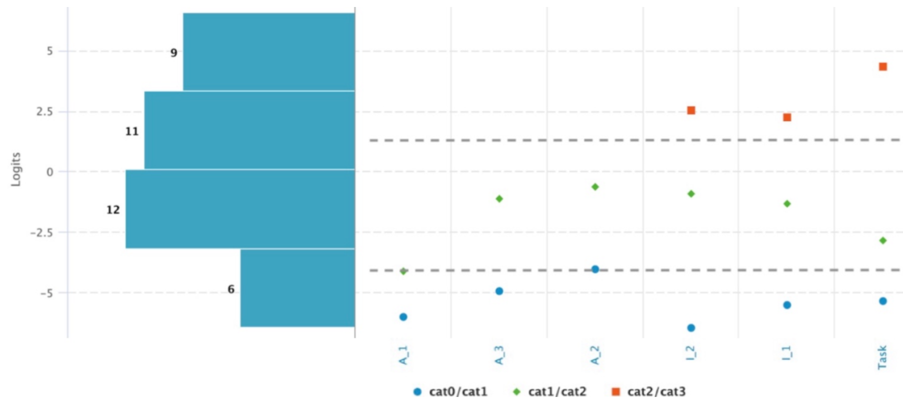

Figure 1: WrightMap by Items

Figure 2 shows, from another angle, that the levels were almost clearly separated and that all items seemed to show an increase of mean location across levels. This reinforces the claim that the proposed levels were distinct with little overlap and that they characterized qualitatively different levels in the continuum of students' understanding-related orientations.

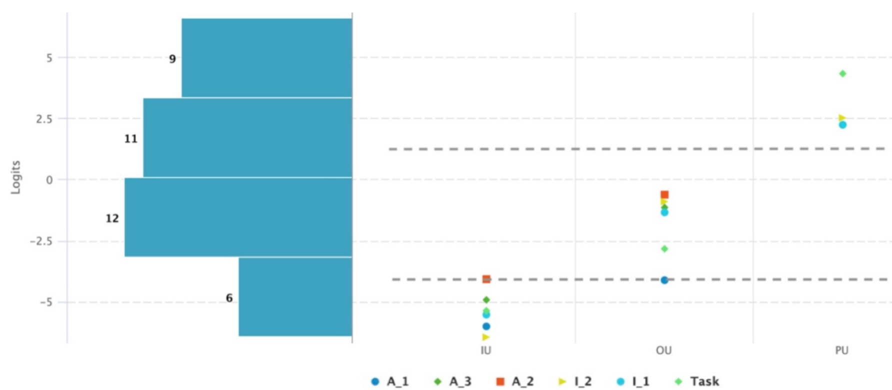

Figure 2: WrightMap by Levels

External Variable Validity. Based on my learning and teaching experiences, I hypothesized that if students felt comfortable learning and doing math, they were more likely to want to understand underlying principles; if they had high levels of math anxiety, they were more likely to resist understanding when it came to learning math. To investigate potential correlations between students' levels of math anxiety and their estimated orientations using the above coding rubric, I created a scatterplot as shown in Figure 3. Results showed that there appeared to be a weak, negative linear correlation between students' level of math anxiety and their estimated orientations with respect to understanding. This suggested that researchers and teachers should take into account (or even better, try to address) students' math anxiety when planning for interventions that

aim to help students prioritize goals and actions that seek to develop deeper understanding of mathematics.

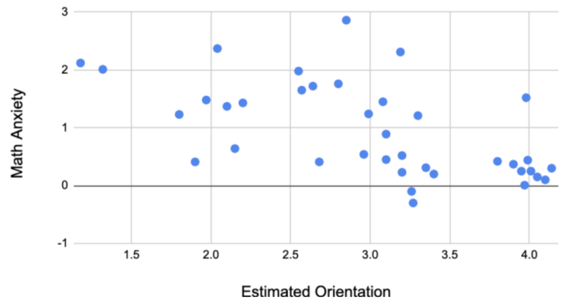

Figure 3: Correlation with Math Anxiety

Concluding Remarks

Results from the above analyses show that the proposed delineation of student orientations and the instrument for measurement had good reliability and validity evidence. This paper provides significant theoretical implications to the field. First, it foregrounds the importance of orientation with respect to understanding in student learning, pushing the field forward into articulating more precisely the hidden differences in students' goals, values, attitudes, and beliefs about understanding and how that impacts students' learning behaviors (and hence learning outcomes). Second, by offering a valid and reliable tool for measuring student orientations through problem solving and interview reflections, the paper paves the way for understanding students' learning and engagement with mathematics more rigorously and comprehensively. Indeed, how intellectually meaningful could it be if students show high levels of engagement in appearance but upon careful examination, they show evidence of readily accepting and applying formulas without bothering to ask why the formulas are true or understand where a particular step comes from? Third, it contributes to a more solid foundation for addressing disparities in different groups of students' intellectual experiences with mathematics and perhaps equivalently importantly, disparities in different groups of students' disciplinary identities and habits of mind.

Practically, the proposed taxonomy and the accompanying coding rubric can serve as a teaching tool for teachers to diagnose students' current states of orientations and model productive, meaningful ways of engaging with mathematics in classrooms at all grade levels. It can also serve as a learning tool for students to self-assess and bootstrap their development of productive learning orientations and mathematical practices. In the spirit of teaching for (Blythe et al., 1998; Newton, 2011; Putnam et al., 1992) and with (Hiebert, 1997) understanding, this study advocates for — besides inviting students to explore, evaluate, reflect, and take risks (as opposed to merely memorize) in expanding their understanding of mathematical and social issues — explicitly supporting students to see the value in and care more about understanding in mathematical learning. With more students orienting towards understanding in the classroom, there is reason to believe that students' learning experiences with school mathematics as both individuals and as a collective will be more positive and productive, contributing to richer, more engaging, and more beautiful mathematical discourses in the classroom and beyond.

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